Chapter 15 Integration

error

1. Given that $\int_{1}^{a}\left(\frac{2}{2x+3}+\frac{3}{3x-1}-\frac{1}{x}\right)dx = ln\,\cancel{1}2.4$ and that $a > 1$, find the value of a. [7]

$$\left[\frac{\cancel{2}\ln(2x+3)}{\cancel{2}} + \frac{\cancel{3}\ln(3x-1)}{\cancel{3}} - \ln x\right]_1^a$$

$$= \left[\ln(2x+3) + \ln(3x-1) - \ln x\right]_1^a$$

$$= \ln(2a+3) + \ln(3a-1) - \ln a - \ln 5 - \ln 2$$

$$= \ln\frac{(2a+3)(3a-1)}{10a}$$

$$2.4 = \frac{(2a+3)(3a-1)}{10a}$$

$$24a = 6a^2 - 2a + 9a - 3$$

$$0 = 6a^2 - 17a - 3$$

$$(a-3)(6a+1) = 0$$

$$a = 3 \quad \text{or} \quad a = -\frac{1}{6}$$

(reject)

2. (a) Show that $\frac{3}{2x-3}+\frac{3}{2x+3}$ can be written as $\frac{12x}{4x^2-9}$.

[2]

$$\frac{3(2x+3)+3(2x-3)}{4x^2-9}$$

$$=\frac{6x+\cancel{9}+6x\cancel{-9}}{4x^2-9}$$

$$=\frac{12x}{4x^2-9}$$

(b) Hence find $\int\frac{12x}{4x^2-9}dx$, giving your answer as a single logarithm and an arbitrary constant.

[3]

$$\int\frac{12x}{4x^2-9}dx=\int\frac{3}{2x-3}+\frac{3}{2x+3}\,dx$$

$$=\frac{3}{2}\ln(2x-3)+\frac{3}{2}\ln(2x+3)+C$$

$$=\frac{3}{2}\ln(2x-3)(2x+3)+C$$

$$=\frac{3}{2}\ln(4x^2-9)+C$$

$$=\ln(4x^2-9)^{3/2}+C$$

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(c) Given that $\int_2^a \frac{12x}{4x^2-9}dx = ln\, 5\sqrt{5}$, where $a > 2$, find the exact value of *a*.

[4]

$$\left[\ln (4x^2-9)^{3/2} \right]_2^a$$

$$= \frac{3}{2} \ln 4a^2-9 - \frac{3}{2} \ln 7$$

$$= \frac{3}{2} \ln \frac{4a^2-9}{7}$$

$$\ln 5\sqrt{5} = \ln \left(\frac{4a^2-9}{7} \right)^{3/2}$$

$$\ln 5^{3/2} = \ln \left(\frac{4a^2-9}{7} \right)^{3/2}$$

$$5 = \frac{4a^2-9}{7}$$

$$35 = 4a^2-9$$

$$a^2 = 11$$

$$a = \sqrt{11}$$

3. A curve is such that $\frac{d^2y}{dx^2} = 5cos\ 2x$. This curve has a gradient of $\frac{3}{4}$ at the point $(-\frac{\pi}{12}, \frac{5\pi}{4})$. Find the equation of this curve.

[8]

$$\frac{dy}{dx} = \frac{5}{2}\sin 2x + C$$

$$\frac{3}{4} = \frac{5}{2}\sin -\frac{\pi}{6} + C$$

$$\frac{3}{4} = -\frac{5}{4} + C$$

$$\frac{8}{4} = C$$

$$C = 2$$

$$\therefore y' = \frac{5}{2}\sin 2x + 2$$

$$y = -\frac{5}{2}\cos 2x \times \frac{1}{2} + 2x + C_2$$

$$= -\frac{5}{4}\cos 2x + 2x + C_2$$

$$\frac{5\pi}{4} = -\frac{5}{4}\cos -\frac{\pi}{6} - \frac{\pi}{6} + C_2$$

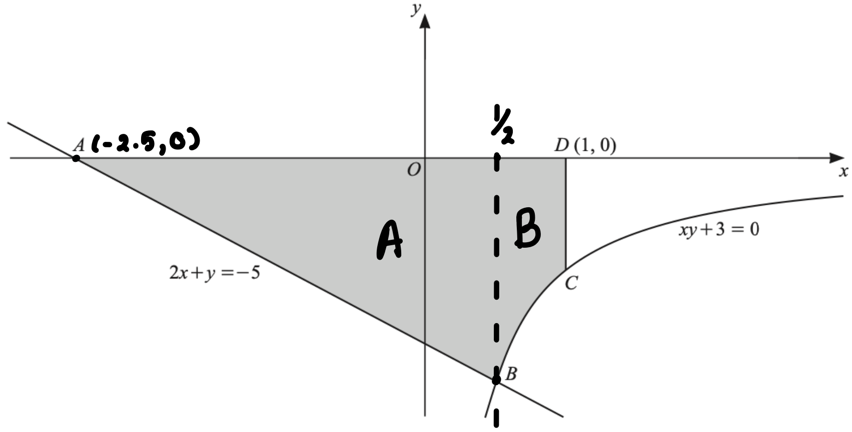
$$\frac{5\pi}{4} = -\frac{5}{4} \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} + C_2$$

$$\frac{5\pi}{4} = -\frac{5\sqrt{3}}{8} - \frac{\pi}{6} + C_2$$

$$C_2 = 5.53$$

$$\therefore y = -\frac{5}{4}\cos 2x + 2x + 5.53$$

4.



The diagram shows the straight line $2x + y = -5$ and part of the curve $xy + 3 = 0$. The straight line intersects the *x*-axis at the point *A* and intersects the curve at the point *B*. The point *C* lies on the curve. The point *D* has coordinates (1, 0). The line *CD* is parallel to the *y*-axis.

(a) Find the coordinates of each of the points *A* and *B*. [3]

$2x + y = -5 \qquad A(x, 0)$

$2x = -5$

$x = -\frac{5}{2} \qquad A\left(-\frac{5}{2}, 0\right)$

$2x + y = -5$

$y = -5 - 2x$

$xy + 3 = 0$

$(-5 - 2x)x + 3 = 0$

$-2x^2 - 5x + 3 = 0$

$2x^2 + 5x - 3 = 0$

$2 \quad -1 \quad 1$

$1 \quad +3 \quad 6$

$(2x - 1)(x + 3) = 0$

$x = \frac{1}{2}$ or $x = -3$ (reject)

$y = -5 - 1$

$= -6$

$B\left(\frac{1}{2}, -6\right)$

(b)Find the area of the shaded region, giving your answer in the form $p + \ln q$, where p and q are positive integers.

[6]

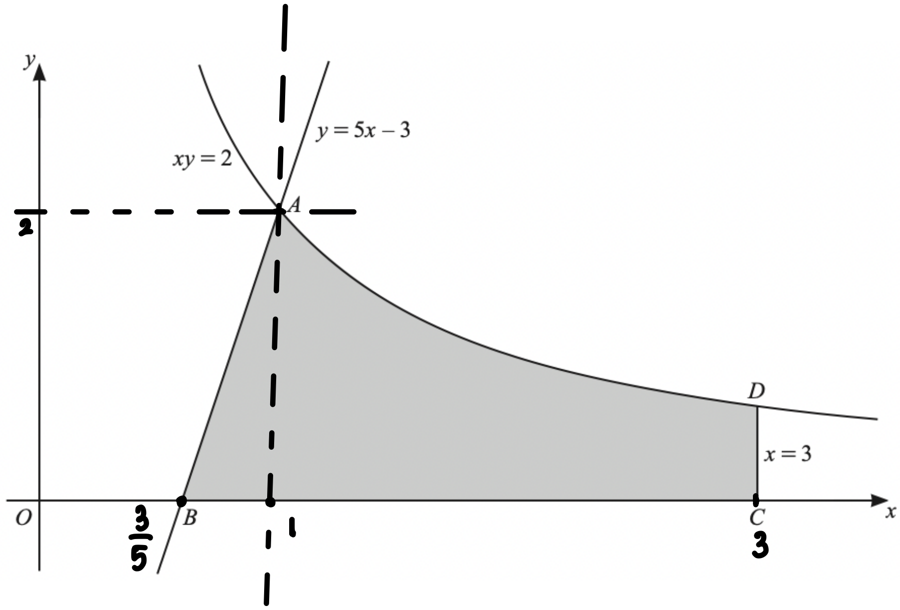
$$\text{Area of } \Delta = \frac{1}{2} \times b \times h$$
$$= \frac{1}{\cancel{2}} \times 3 \times \cancel{6}\,3$$
$$= 9$$

$$xy + 3 = 0$$
$$xy = -3$$
$$y = -\frac{3}{x}$$

$$\int_{\frac{1}{2}}^{1} -\frac{3}{x}\,dx = \left[-3\ln x\right]_{\frac{1}{2}}^{1}$$
$$= +3\ln\frac{1}{2} = 3\ln 2^{-1}$$
$$= |-3\ln 2| \leftarrow \text{cause Area (only positive)}$$
$$= 3\ln 2$$

$$\text{shaded Area} = 9 + 3\ln 2$$
$$= 9 + \ln 8$$

5.



The diagram shows part of the curve $xy = 2$ intersecting the straight line $y = 5x - 3$ at the point A. The straight line meets the x-axis at the point B. The point C lies on the x-axis and the point D lies on the curve such that the line CD has equation $x = 3$. Find the exact area of the shaded region, giving your answer in the form $p + \ln q$, where p and q are constants.

[8]

$xy = 2$

$x(5x-3) = 2$

$5x^2 - 3x = 2$

$5x^2 - 3x - 2 = 0$

$(x-1)(5x+2) = 0$

$x = 1$ or $x = -\frac{2}{5}$ (reject)

$y = 2$

$A(1,2)$

$y = 5x - 3$

$y = 0,\ 0 = 5x - 3$

$x = \frac{3}{5}$ $\quad B(3/5, 0)$

Shaded Area $= \frac{1}{2}bh + \int_1^3 2/x \, dx$

$= \frac{1}{\cancel{2}} \times \cancel{2} \times \frac{2}{5} + \left[2\ln x\right]_1^3$

$= \frac{2}{5} + 2\ln 3 - 2\cancel{\ln 1}^{\,0}$

$= 2/5 + \ln 9$

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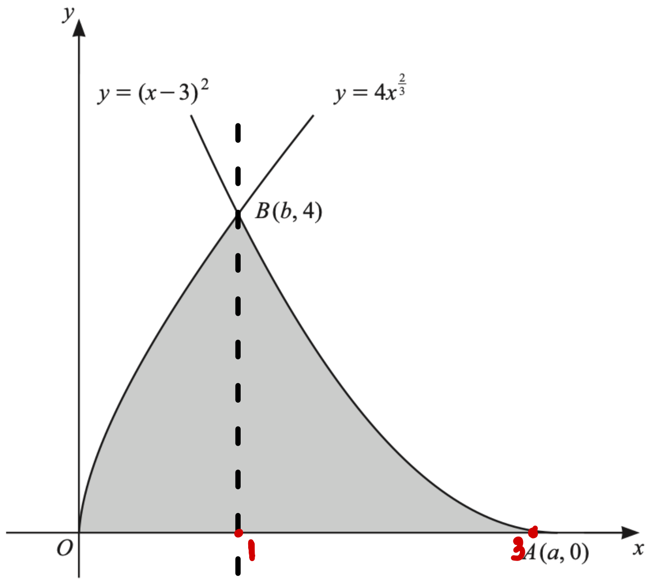
Additional working space for question 5..

6.



The diagram shows part of the graphs of $y = 4x^{\frac{2}{3}}$ and $y = (x - 3)^2$. The graph of $y = (x - 3)^2$ meets the x-axis at the point $A(a, 0)$ and the two graphs intersect at the point $B(b, 4)$.

a. Find the value of a and of b. [2]

$y=(x-3)^2 \quad A(a,0)$

$0 = x-3$

$x = 3 \qquad a=3$

$B(b,4)$

$4=(b-3)^2$

$\pm 2 = b-3$

$2 = b-3$ or $-2=b-3$

$b=5 \qquad b=1$

(reject) $\qquad B(1,4)$

b. Find the area of the shaded region. [5]

$x^2 - 6x + 9$

Shaded Area $= \int_0^1 4x^{2/3}\,dx + \int_1^3 (x-3)^2\,dx$

$= \left[4x^{5/3} \times \frac{3}{5}\right]_0^1 + \left[\frac{x^3}{3} - \frac{\cancel{6}^{\,3}x^2}{\cancel{2}} + 9x\right]_1^3$

$= \frac{12}{5} + \cancel{9} - \cancel{27} + \cancel{27} - \frac{1}{3} + 3 - \cancel{9}$

$= \frac{76}{15}$

7. Giving your answer in its simplest form, find the exact value of

a. $\int_{0}^{4} \frac{10}{5x+2} dx$,

[4]

$$\left[2\ln(5x+2) \right]_0^4$$

$$= 2\ln 22 - 2\ln 2$$

$$= 2\ln 11$$

$$= \ln 121$$

b. $\int_{0}^{ln2} (e^{4x+2})^2 dx$.

[5]

$$\int_0^{\ln 2} e^{8x+4} \, dx$$

$$= \left[\frac{e^{8x+4}}{8} \right]_0^{\ln 2}$$

$$= \frac{1}{8} e^{\ln 2^8 + 4} - \frac{1}{8} e^4$$

$$= \frac{1}{8} e^{\ln 2^8} \times e^4 - \frac{1}{8} e^4$$

$$= \frac{1}{8} e^4 (2^8 - 1)$$

$$= \frac{255}{8} e^4$$

8. (a) (i) Given that $f(x) = \frac{1}{\cos x}$, show that $f'(x) = \tan x \sec x$.

[3]

$f(x) = (\cos x)^{-1}$

$f'(x) = -(\cos x)^{-2} \times -\sin x$

$= \frac{1}{\cos^2 x} \times \sin x = \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$

$= \tan x \sec x$

(shown)

(ii) Hence find $\int (3 \tan x \sec x - \sqrt[4]{e^{3x}}) dx$.

[3]

$\int 3 \tan x \sec x \, dx - \int e^{3/4 x} dx$

$= \frac{3}{\cos x} - \frac{4}{3} e^{3/4 x} + C$

(b) Given that $\int_{2}^{5} \frac{p}{px+10} dx = \ln 2$, find the value of the positive constant *p*.

[5]

$$\left[\frac{\cancel{p} \ln(px+10)}{\cancel{p}} \right]_2^5 = \ln 2$$

$$\ln(5p+10) - \ln(2p+10) = \ln 2$$

$$\ln \frac{5p+10}{2p+10} = \ln 2$$

$$5p+10 = 4p+20$$

$$p = 10$$

9. (a) Given that $\int_1^a (\frac{1}{x} - \frac{1}{2x+3})dx = ln\,3$, where $a > 0$, find the exact value of a, giving your answer in simplest surd form.

[6]

$$\left[\ln x - \frac{1}{2}\ln(2x+3)\right]_1^a = \ln 3$$

$$\ln a - \frac{1}{2}\ln(2a+3) + \frac{1}{2}\ln 5 = \ln 3$$

$$\ln a - \ln\left(\frac{2a+3}{5}\right)^{1/2} = \ln 3$$

$$\ln \frac{a}{\left(\frac{2a+3}{5}\right)^{1/2}} = \ln 3$$

$$\frac{a}{3} = \left(\frac{2a+3}{5}\right)^{1/2}$$

$$\frac{a^2}{9} = \frac{2a+3}{5}$$

$$5a^2 = 18a + 27$$

$$5a^2 - 18a - 27 = 0$$

$$a = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{18 + \sqrt{324 + 540}}{10}$$

$$= \frac{18 + 12\sqrt{6}}{10} = \frac{9 + 6\sqrt{6}}{5}$$

(b) Find the exact value of $\int_{0}^{\frac{\pi}{3}} (sin\,(2x + \frac{\pi}{3}) - 1 + cos\,2x)dx$.

[5]

$$\left[-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right) - x + \frac{1}{2}\sin 2x\right]_{0}^{\pi/3}$$

$$= -\frac{1}{2}\overset{-1}{\cancel{\cos\pi}} - \frac{\pi}{3} + \frac{1}{2}\underset{\frac{\sqrt{3}}{2}}{\cancel{\sin\frac{2\pi}{3}}} + \frac{1}{2}\cancel{\cos\frac{\pi}{3}}\ \frac{1}{2}$$

$$= \frac{1}{2} - \frac{\pi}{3} + \frac{\sqrt{3}}{4} + \frac{1}{4}$$

$$= \frac{6 - 4\pi + 3\sqrt{3} + 3}{12}$$

$$= \frac{9}{12} - \frac{4\pi}{12} + \frac{3\sqrt{3}}{12}$$

$$= \frac{3}{4} - \frac{\pi}{3} + \frac{\sqrt{3}}{4}$$

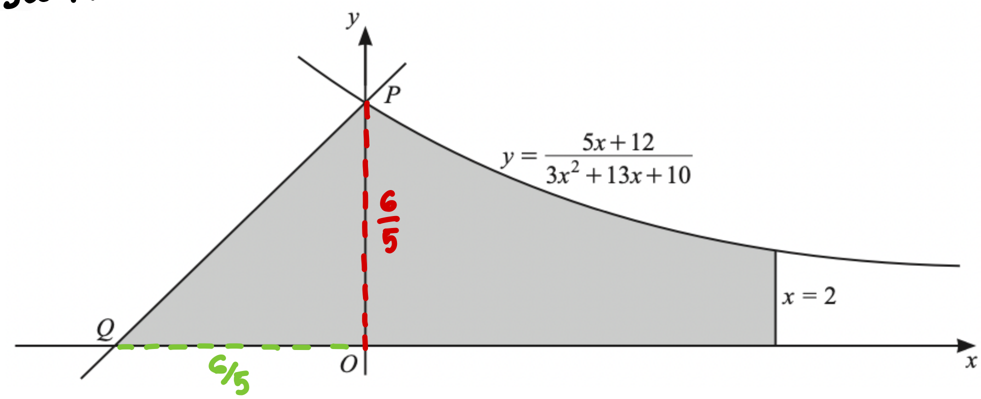
10. (a) Show that $\frac{1}{x+1} + \frac{2}{3x+10}$ can be written as $\frac{5x+12}{3x^2+13x+10}$.

[1]

$$\frac{3x+10+2x+2}{(x+1)(3x+10)}$$

$$= \frac{5x+12}{3x^2+10x+3x+10} = \frac{5x+12}{3x^2+13x+10} \text{ (shown)}$$

(b)



The diagram shows part of the curve $y = \frac{5x+12}{3x^2+13x+10}$, the line $x = 2$ and a straight line of gradient 1.The curve intersect the y-axis at the point P.The line of gradient 1 passes through P and intersects the x-axis at the point Q.Find the area of the shaded region, giving your answer in the form $a + \frac{2}{3} ln\, (b\sqrt{3})$, where *a* and *b* are constants.

[9]

$$y = \frac{5x+12}{3x^2+13x+10}$$

$$x = 0,\ y = \frac{12}{10} = \frac{6}{5} \qquad P(0, 6/5)$$

$$m = \frac{6/5 - 0}{0 - x}$$

$$-x = \frac{6}{5}$$

$$x = -\frac{6}{5} \qquad Q(-6/5, 0)$$

Area of $\triangle = \frac{1}{2}bh$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{6}^{3}}{5} \times \frac{6}{5} = \frac{18}{25}$$

Additional working space for question 10

$$\text{Shaded Area} = \frac{18}{25} + \int_0^2 \frac{5x+12}{3x^2+13x+10}\,dx$$

$$= \frac{18}{25} + \int_0^2 \frac{1}{x+1} + \frac{2}{3x+10}\,dx$$

$$= \frac{18}{25} + \left[\ln(x+1) + \frac{2}{3}\ln(3x+10) \right]_0^2$$

$$= \frac{18}{25} + \ln 3 + \frac{2}{3}\ln 16 - \frac{2}{3}\ln 10$$

$$= \frac{18}{25} + \ln 3 + \frac{2}{3}\ln\frac{8}{5}$$

$$= \frac{18}{25} + \ln\left(3^{3/2}\right)^{2/3} + \frac{2}{3}\ln\frac{8}{5}$$

$$= \frac{18}{25} + \frac{2}{3}\ln 3^{3/2} + \frac{2}{3}\ln\frac{8}{5}$$

$$= \frac{18}{25} + \frac{2}{3}\ln\frac{8\times 3\sqrt{3}}{5}$$

$$= \frac{18}{25} + \frac{2}{3}\ln\frac{24\sqrt{3}}{5}$$

11. The gradient of the normal to a curve at the point (x, y) is given by $\frac{x}{x+1}$.

a. Given that the curve passes through the point (1, 4), show that its equation is $y = 5 - lnx - x$.

[5]

$$\frac{dy}{dx} = -\left(\frac{x+1}{x}\right)$$

$$= -1 - \frac{1}{x}$$

$$y = -x - \ln x + C$$

$$4 = -1 + C$$

$$C = 5$$

$$\therefore y = 5 - \ln x - x \text{ (shown)}$$

b. Find, in the form $y = mx + c$, the equation of the tangent to the curve at the point where $x = 3$.

[3]

$y = 5 - x - \ln x$

$x = 3,\ y = 5 - 3 - \ln 3$

$= 2 - \ln 3$

$y' = -1 - \frac{1}{x}$

$= -1 - \frac{1}{3} = -\frac{4}{3}$

$y = -\frac{4}{3}x + c$

$2 - \ln 3 = -4 + c$

$6 - \ln 3 = c$

$y = -\frac{4}{3}x + 6 - \ln 3$

12. Find the exact value of $\int_{2}^{4} \frac{(x+1)^2}{x^2} dx$.

[6]

$$\int_2^4 \frac{x^2+2x+1}{x^2}\, dx$$

$$= \int_2^4 1 + 2x^{-1} + x^{-2}\, dx$$

$$= \left[x + 2\ln x - x^{-1} \right]_2^4$$

$$= 4 + \ln 16 - \frac{1}{4} - 2 - \ln 4 + \frac{1}{2}$$

$$= \frac{9}{4} + \ln 4$$

$$= \frac{9}{4} + 2\ln 2$$

13. A curve has equation $y = x\cos x$.

a. Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \cos x - x\sin x$$

[2]

b. Find the equation of the normal to the curve at the point where $x = \pi$, giving your answer in the form $y = mx+c$.

[4]

$m = \cos x - x\sin x$

$= -1$

$m_{normal} = 1$

$y = x + c$

$y = x\cos x$

$x = \pi, y = -\pi$

$-\pi = \pi + c$

$c = -2\pi$

$\therefore y = x - 2\pi$

c. Using your answer to **part (a)**, find the exact value of $\int_0^{\frac{\pi}{6}} x \sin x \, dx$.

[5]

$$\frac{d}{dx} x\cos x = \cos x - x\sin x$$

$$x\sin x = \cos x - \frac{d}{dx} x\cos x$$

$$\int x\sin x \, dx = \int \cos x \, dx - \int \frac{d}{dx} x\cos x \, dx$$

$$= \sin x - x\cos x$$

$$\int_0^{\pi/6} x\sin x \, dx = \Big[\sin x - x\cos x\Big]_0^{\pi/6}$$

$$= \cancel{\sin\frac{\pi}{6}}^{\,1/2} - \frac{\pi}{6}\cancel{\cos\frac{\pi}{6}}^{\,\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} - \frac{\sqrt{3}\pi}{12}$$

14. The equation of a curve is $y = x\sqrt{16 - x^2}$ for $0 \leq x \leq 4$.

a. Find the exact coordinates of the stationary point of the curve. [6]

$$\frac{dy}{dx} = (16-x^2)^{1/2} + x \times \frac{1}{\cancel{2}}(16-x^2)^{-1/2} \times -\cancel{2}x$$

$$= (16-x^2)^{1/2} - x^2(16-x^2)^{-1/2}$$

$$= \frac{16-x^2-x^2}{(16-x^2)^{1/2}}$$

$$= \frac{-2x^2+16}{\sqrt{16-x^2}}$$

$$\frac{dy}{dx} = 0$$

$$-2x^2+16=0$$

$$2x^2 = 16$$

$$x^2 = 8$$

$$x = 2\sqrt{2}$$

$$y = 2\sqrt{2}\,\sqrt{16-8}$$

$$= \sqrt{8} \times \sqrt{8}$$

$$= 8$$

$$(2\sqrt{2}, 8)$$

b. Find $\frac{d}{dx}(16-x^2)^{\frac{3}{2}}$ and hence evaluate the area enclosed by the curve $y = x\sqrt{16-x^2}$ and the line $y = 0, x = 1$ and $x = 3$. [5]

$$\frac{d}{dx}(16-x^2)^{3/2} = \frac{3}{\cancel{2}}(16-x^2)^{1/2} \times -\cancel{2}x$$

$$= -3x(16-x^2)^{1/2}$$

$$\frac{d}{dx}(16-x^2)^{3/2} = -3x\sqrt{(16-x^2)}$$

$$-\frac{1}{3}\frac{d}{dx}(16-x^2)^{3/2} = x\sqrt{16-x^2}$$

$$\int_1^3 x\sqrt{16-x^2}\,dx = \left[-\frac{1}{3}(16-x^2)^{3/2}\right]_1^3$$

$$= -\frac{1}{3}(16-9)^{3/2} + \frac{1}{3}(16-1)^{3/2}$$

$$= 13.2$$